INTERVAL-VALUED FUZZY SET MODELLING OF SYSTEM RELIABILITY

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System reliability modeling in terms of fuzzy set theory is basically utilizing the Type I fuzzy sets, where the fuzzy membership is assumed as point-wise positive function ranging on [0,1]. Such a practice might not be practical because an interval-valued membership may reflect the vagueness of system better according to human thinking patterns. In this paper, we explore the basics of the interval-valued fuzzy sets theory and illustrate its application in terms of an industrial example.

1 Introduction

System operating and maintenance data are often imprecise and vague. Therefore fuzzy sets theory (Zadeh, [7]) opened the way for facilitating the modeling fuzziness aspect of system reliability. A fundamental issue is the treatment of membership function because fuzzy set as an extension of classical set in terms of extending the {0,1} two-valued indicator function characterizing a crisp set into a membership function ranging on interval [0,1] which characterizes a fuzzy set. Most of the fuzzy reliability modeling efforts is assuming a membership function, which could be regarded as a *point* estimate of the degree of belief of belongingness relation, for the reflection of vague nature of system operating and maintenance data. However, it may be more logical and practical to assume an interval-valued membership grade, which could be regarded as an interval-valued estimate of the degree of belief of the subordination relation because as a general and natural human thinking pattern, the degree of fuzziness appears as an interval-valued number on [0,1]. In other words, it is natural to use a special class of type II fuzzy sets - interval-valued fuzzy set (IVFS) (Zahed, [8]) to describe the fuzzy aspect of system reliability.

Section 2 contains the elementary concept and operations on IVFSs. Furthermore, the relation between IVFS and rough set (Pawlak, [5]) and thus the IVFS decomposition theorem is established in section 3. In Section 4, the probability of IVFS is defined. In Section 4 a stress-tension style reliability model is proposed for analyzing the state of a repairable system. Section 5 is used to illustrate reliability analysis details in terms of an industrial example - cement roller data (Love and Guo, [4]). Section 6 gives a few comments on the IVFS system reliability analysis.

2 Interval-Valued Fuzzy Set

1.1. *Concept of Interval-Valued Fuzzy Sets*

Definition 1. A closed interval $\bar{a} \triangleq [a^l, a^u], a^l, a^u \in \mathbb{R}$, and $a^l \leq a^u$ is called real-valued interval number. If $a^l, a^u \in [0,1]$, $[a^l, a^u]$ is called an interval number on unit interval or simply interval number. Let $\mathbb{I}[0,1] \triangleq \{\bar{a} = [a^l, a^u] \mid 0 \leq a^l \leq a^u \leq 1\}$ then it is the collection of all interval numbers (on unit interval [0,1]).

Definition 2. Let set *U* denote a discourse. An interval-valued fuzzy set (IVFS) $\vec{A}$ is a mapping from *U* to $\mathbb{I}[0,1]$:

$$\bar{\mu}_{\vec{A}} : U \rightarrow \mathbb{I}[0,1] \tag{1}$$

For $\forall u \in U$,

$$\bar{\mu}_{\vec{A}}(u) \triangleq \left[\mu^l_{\vec{A}}(u), \mu^u_{\vec{A}}(u)\right] \tag{2}$$

where,

$$\mu^l_{\vec{A}} : U \rightarrow [0,1] \quad and \quad \mu^u_{\vec{A}} : U \rightarrow [0,1] \tag{3}$$

such that,

$$0 \leq \mu^l_{\vec{A}}(u) \leq \mu^u_{\vec{A}}(u) \leq 1 \quad for \quad \forall u \in U \tag{4}$$

Therefore, an IVFS is characterized by an interval-valued membership function $\bar{\mu}_{\vec{A}}$, denoted as,

$$\vec{A} = \left\{ \left\langle u, \bar{\mu}_{\vec{A}}(u) \right\rangle \mid u \in U \right\} \tag{5}$$

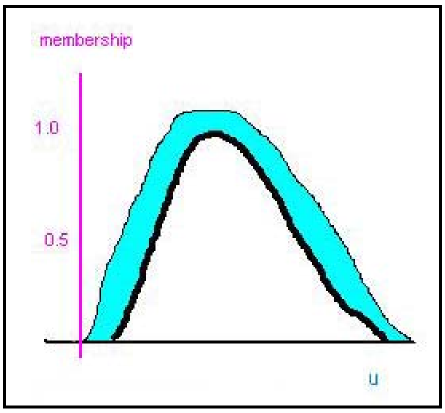

Figure 1. Membership of an IVFS.

Atanassov [2] proposed concept of intuitionstic fuzzy set (IFS), which is equivalent to an IVFS. Mapping

$$\pi(u) \triangleq 1 - \mu_{\underset{\sim}{A}}(u) - v_{\underset{\sim}{A}}(u) \tag{6}$$

defines the depth of the degree of vague uncertainty of an IVFS and therefore,

$$\pi \triangleq \mu_{\tilde{A}}^{u} - \mu_{\tilde{A}}^{l} \tag{7}$$

1.2. *A Geometric Interpretation*

Agustench, Bustince and Mohedano [1] gave a geometric interpretation of an IVFS, which clearly identifies the triangle OAB (red-colored) within the unit-cube under the coordinate system $\left(\mu_{\tilde{A}}^{l}, \mu_{\tilde{A}}^{u}, \pi\right)$ (i.e., lower-membership $\mu_{\tilde{A}}^{l}$ - horizontal axis (purple-colored), upper membership grade $\mu_{\tilde{A}}^{u}$ - green-colored axis, the depth of vagueness $\pi = \mu_{\tilde{A}}^{u} - \mu_{\tilde{A}}^{l}$ - vertical axis) the projection space. In other words, an IVFS is a mapping from *U* to the triangle OAB.

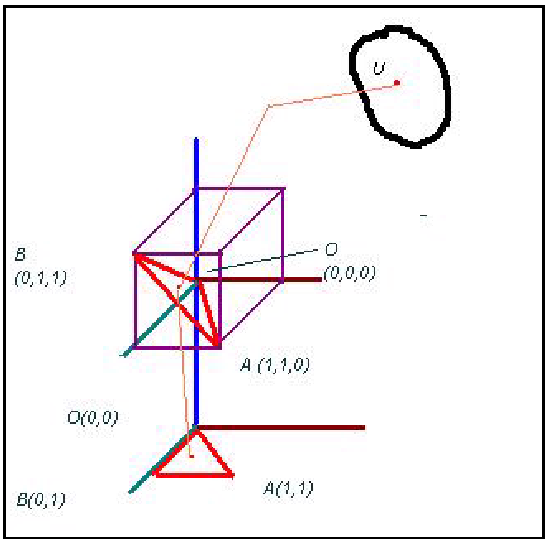

Figure 2. Geometric interpretation of an IVFS.

If we only look at $\left(\mu_{\tilde{A}}^{l}(u), \mu_{\tilde{A}}^{u}(u)\right)$, then a curve defined inside the triangle OAB [(0,0),(1,1),(0,1)] on the bottom will characterize an IVFS. It is needless to say that the geometric interpretation should help us a better understanding and thus specification of an IVFS in practice.

1.3. *Basic Operations on IVFSs*

Let $\overleftrightarrow{A},\overleftrightarrow{B}\in\mathcal{F}_{\mathbb{I}}(U)$ be two interval fuzzy sets on discourse *U*. The three basic operations: union, intersection and complement operations of interval fuzzy sets $\overleftrightarrow{A},\overleftrightarrow{B}$, are defined as:

(i) Union of $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$:

$$\left.\begin{aligned}&\overleftrightarrow{A}\cup\overleftrightarrow{B}=\left\{u,\overline{\mu}_{\overleftrightarrow{A}\cup\overleftrightarrow{B}}(u)\middle|u\in U\right\}\\&\overline{\mu}_{\overleftrightarrow{A}\cup\overleftrightarrow{B}}(u)\triangleq\overline{\mu}_{\overleftrightarrow{A}}(u)\vee\overline{\mu}_{\overleftrightarrow{B}}(u)\\&=\left[\mu^l_{\overleftrightarrow{A}}(u)\vee\mu^l_{\overleftrightarrow{B}}(u),\mu^u_{\overleftrightarrow{A}}(u)\vee\mu^u_{\overleftrightarrow{B}}(u)\right]\end{aligned}\right\}\tag{8}$$

(ii) Intersection of $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$:

$$\left.\begin{aligned}&\overleftrightarrow{A}\cap\overleftrightarrow{B}=\left\{u,\overline{\mu}_{\overleftrightarrow{A}\cap\overleftrightarrow{B}}(u)\middle|u\in U\right\}\\&\overline{\mu}_{\overleftrightarrow{A}\cap\overleftrightarrow{B}}(u)\triangleq\overline{\mu}_{\overleftrightarrow{A}}(u)\wedge\overline{\mu}_{\overleftrightarrow{B}}(u)\\&=\left[\mu^l_{\overleftrightarrow{A}}(u)\wedge\mu^l_{\overleftrightarrow{B}}(u),\mu^u_{\overleftrightarrow{A}}(u)\wedge\mu^u_{\overleftrightarrow{B}}(u)\right]\end{aligned}\right\}\tag{9}$$

(iii) Complement of $\overleftrightarrow{A}$:

$$\left.\begin{aligned}&\overleftrightarrow{A}^c=\left\{u,\overline{\mu}_{\overleftrightarrow{A}^c}(u)\middle|u\in U\right\}\\&\overline{\mu}_{\overleftrightarrow{A}^c}(u)\triangleq\left[1-\overline{\mu}^u_{\overleftrightarrow{A}}(u),1-\overline{\mu}^l_{\overleftrightarrow{A}}(u)\right]\end{aligned}\right\}\tag{10}$$

Other operations, say, *t*-norm and *t*-conorm will not be mentioned here for briefness but are critical in IVFS inferences.

3 Decomposition of an IVFS

The critical role of fuzzy set decomposition in fuzzy mathematical theory is that it links a fuzzy set to the common (crisp) set. For the Type I fuzzy set case the decomposition takes a form of

$$\tilde{A}=\bigcup_{\lambda\in[0,1]}\left[\lambda\bullet A_\lambda\right]\tag{11}$$

where

$$A_\lambda=\left\{u\in U\middle|\mu_{\tilde{A}}(u)\geq\lambda\right\},\quad\forall\lambda\in[0,1]\tag{12}$$

The key issue here is that in the case of IVFS, the membership is an interval $\overline{a}\in\mathbb{I}[0,1]$. Therefore, the decomposition should not be performed by *line-cut* (Type I fuzzy set) but by an *interval-cut*, in other words, it is necessary to investigate the set

$$\left\{\overline{\mu}_{\overleftrightarrow{A}}(u)\geq\overline{\lambda}\right\},\quad\forall\overline{\lambda}=\left[\lambda^l,\lambda^u\right]\in\mathbb{I}[0,1]\tag{13}$$

that is,

$$\left\{u \in U \middle| \mu_{\tilde{A}^l}(u) \le \lambda^l\right\} \tag{14}$$

and,

$$\left\{u \in U \middle| \mu_{\tilde{A}^u}(u) \le \lambda^u\right\} \tag{15}$$

Therefore, the λ^l -level cut set $A_{\lambda^l} = \left\{u^l \in U \middle| \mu_{\tilde{A}^l}(u^l) \le \lambda^l\right\}$ and λ^u -level cut set $A_{\lambda^u} = \left\{u^u \in U \middle| \mu_{\tilde{A}^u}(u^u) \le \lambda^u\right\}$ can be used to characterized the interval-valued $\overline{\lambda}$ -cut set, denoted as:

$$A_{\overline{\lambda}} \triangleq \left\langle A_{\lambda^l}, A_{\lambda^u} \right\rangle, \quad \forall \overline{\lambda} = \left[\lambda^l, \lambda^u\right] \in \mathbb{I}[0,1] \tag{16}$$

Theorem: An interval-valued fuzzy set $\overleftrightarrow{A}$ can be represented as:

$$\overleftrightarrow{A} = \bigcup_{\overline{\lambda} \in \mathbb{I}[0,1]} \overline{\lambda} \bullet A_{\overline{\lambda}} \tag{17}$$

where

$$\overline{\lambda} \bullet A_{\overline{\lambda}} \triangleq \left[\lambda^l \bullet A_{\lambda^l}, \lambda^u \bullet A_{\lambda^u}\right] \tag{18}$$

Proof: In terms of the construction definitions,

$$\overleftrightarrow{A} = \bigcup_{\overline{\lambda} \in \mathbb{I}[0,1]} \left[\lambda^l \bullet A_{\lambda^l}, \lambda^u \bullet A_{\lambda^u}\right] = \left\langle \bigcup_{\lambda^l \in [0,1]} \lambda^l \bullet A_{\lambda^l}, \bigcup_{\lambda^u \in [0,1]} \lambda^u \bullet A_{\lambda^u} \right\rangle = \left\langle \tilde{A}^l, \tilde{A}^u \right\rangle \tag{19}$$

It is obvious that the interval A_{λ^l} can be treated as an lower approximation to the set A_{λ} and the interval A_{λ^u} can be treated as the upper approximation to the set A_{λ}. Notice that $A_{\lambda^l} \subseteq A_{\lambda} \subseteq A_{\lambda^u}$. It is reasonable to argue that the $\overline{\lambda}$ -cut sets induces rough set in the sense of Pawlak [5]. This linkage may also promote a better understanding of the concept of an IVFS and even help to specify the interval-valued membership more intuitively.

4 The Probability of an IVFS

The probability of (Type I) fuzzy event $\tilde{A}$ on *U*, $\tilde{A} \in \mathcal{F}(U)$, is,

$$\Pr\left[\tilde{A}\right] = E_P\left[\mu_{\tilde{A}}(X)\right] = \int_U \mu_{\tilde{A}}(u)\, dP(u) \tag{20}$$

In the context of IVFS, the relation between the interval-valued membership $\overline{\mu}_{\tilde{A}}$ and the probability of the interval-valued fuzzy set will maintain a similar form:

$$\Pr\left[\vec{\tilde{A}}\right] = E_P\left[\bar{\mu}_{\tilde{A}}(X)\right] = \int_U \bar{\mu}_{\tilde{A}}(u)\,dP(u) = \left[\Pr\left[\tilde{A}^l\right], \Pr\left[\tilde{A}^u\right]\right] = \left[p^l_{\tilde{A}}, p^u_{\tilde{A}}\right] \quad (21)$$

This expression will give a probability interval for the IVFS $\vec{\tilde{A}}$.

5 An IVFS Reliability Model of Repairable Systems

5.1. *A Virtual Allowable Capacity Model for Repairable System*

A basic idea of the reliability model proposed here is essentially taking from that of the traditional stress-tension modelling of an engineering structure. If we treat a repairable system as a virtual engineering structure, then the system parameters, the maintenance parameters and its operational environment parameters together can form a *virtual allowable capacity*, denoted as C_a, which would restrain or control the system functioning state. The virtual allowable capacity plays a role similar to the stress level in the stress-tension model, which will determine a virtual allowable operating time, denoted as T_a. On the other hand, the system functioning or operating causes system wear-out and increases its failure hazard. Therefore, the actual system functioning plays a role similar to the tension level, denoted as T.

The limiting state equation of the reliability of functioning system is:

$$Z = T_a - T \quad (22)$$

Furthermore it is assume that the limiting state Z is normally distributed random variable. It is intuitive to say that both T_a and T are random and fuzzy in nature. The failure of the system is assumed to be an interval-valued fuzzy event with membership function:

$$\bar{\mu}_{\tilde{A}}(z) = \left[\mu_{\tilde{A}^l}(z), \mu_{\tilde{A}^u}(z)\right] \quad (23)$$

5.2. *The Cement Roller Example*

A set industrial data, e.g., Love and Guo [4]-a set of operating data extracted from a Canadian cement plant is used. Guo and Love [3] performed a fuzzy analysis on the same data in terms of fuzzy logical function method for obtaining the point-wise relative membership grades $\mu^r_{C_a}(u)$. For illustration purpose, we convert $\mu^r_{C_a}(u)$ into interval-valued memship grades $\bar{\mu}_{\tilde{C}_a}(t_a)$ by assigning the depth of vagueness π=0.1 at $\mu^r_{C_a}(u)$=0.5 and π=0 at $\mu^r_{C_a}(u)$=0 or 1.0.

For a recorded failure (or PM) time, the corresponding the allowable time satisfies

$$\bar{\mu}_{\tilde{C}_a}(t) = \bar{1} - \frac{1}{t_{\max}}\bar{t}_a \quad (24)$$

that is, the allowable time

$$\overline{t_a} = t_{\max}\left(\overline{1} - \overline{\mu}_{\tilde{C}_a}\left(t\right)\right) \tag{25}$$

Therefore the virtual system state:

$$\overline{z} = \overline{t_a} - \overline{t} \tag{26}$$

For failure times, $t_{max}=max\{t_1\kappa_1,\ldots, t_{31}\kappa_{31}\}=147$, while for the censoring (PM) times, $t_{max}=max\{t_1(1-\kappa_1),\ldots, t_{31}(1-\kappa_{31})\}=217$. Then $\overline{\mu}_{C_a}\left(u\right)$, $\overline{t_a}$ and $\overline{z}$ interval-values are calculated and listed in Table 1.

Table 1. "Observed" $\overline{t_a}$ and $\overline{z}$ -valued for each PM.

t_i	κ_i	$\mu^r_{C_a}\left(u\right)$	$\overline{\mu}_{C_a}\left(u\right)$	$\overline{t_a}$	$\overline{z}$
54	0	0.5	[0.450,0.550]	[97.65,119.35]	[43.65,65.35]
133	1	0.8	[0.780,0.820]	[26.46,32.34]	[-106.54,-100.66]
147	0	0.818	[0.800,0.836]	[35.588,43.4]	[-111.412,-103.6]
72	1	0.6	[0.560,0.640]	[52.92,64.68]	[-19.08,-7.32]
105	1	0.8	[0.780,0.820]	[26.46,32.34]	[-78.54,-72.66]
115	0	0.375	[0.338,0.413]	[127.379,143.654]	[12.379,28.654]
141	0	0.538	[0.492,0.584]	[90.272,110.236]	[-50.728,-30.764]
59	1	0.667	[0.630,0.701]	[43.953,54.39]	[-15.047,-4.61]
107	0	0.125	[0.113,0.138]	[187.054,192.479]	[80.054,85.479]
59	0	0.2	[0.180,0.220]	[169.26,177.94]	[110.26,118.94]
36	1	0.4	[0.360,0.440]	[82.32,94.08]	[46.32,58.08]
210	0	0	[0.000,0.000]	[217,217]	[7,7]
45	1	0.429	[0.386,0.472]	[77.616,90.258]	[32.616,45.258]
69	0	0.6	[0.560,0.640]	[78.12,95.48]	[9.12,26.48]
55	0	0.889	[0.877,0.900]	[21.7,26.691]	[-33.3,-28.309]
74	1	0.875	[0.853,0.888]	[16.464,21.609]	[-57.536,-52.391]
124	1	0.778	[0.756,0.800]	[29.4,35.868]	[-94.6,-88.132]
147	1	0.667	[0.630,0.701]	[44.1,53.655]	[-102.9,-93.345]
171	0	0.375	[0.338,0.413]	[127.379,143.654]	[-43.621,-27.346]
40	1	0.667	[0.630,0.701]	[43.953,54.39]	[3.953,14.39]
77	1	0.778	[0.756,0.800]	[29.4,35.868]	[-47.6,-41.132]
98	1	0.6	[0.560,0.640]	[52.92,64.68]	[-45.08,-33.32]
108	1	0.6	[0.560,0.640]	[52.92,64.68]	[-45.08,-33.32]
110	0	0.667	[0.630,0.701]	[64.883,80.29]	[-45.117,-29.71]
85	1	1	[1.000,1.000]	[0,0]	[-85,-85]
100	1	0.556	[0.512,0.600]	[58.8,71.736]	[-41.2,-28.264]
115	1	0.8	[0.780,0.820]	[26.46,32.34]	[-88.54,-82.66]
217	0	0.2	[0.180,0.220]	[169.26,177.94]	[-47.74,-39.06]

25	1	0.429	[0.386,0.472]	[77.616,90.258]	[52.616,65.258]
50	1	0.429	[0.386,0.472]	[77.616,90.258]	[27.616,40.258]

From the table, it is easy to notice that most of the failure cases (κ_i=1), the $\overline{z}$ -values observed are negative, which indicates the system falls in failure and damaged state, while quite a few of the censoring cases, the $\overline{z}$ -values observed are positive, which indicates the system is still in "reliable" and "safe" state. The signs of these "observed" $\overline{z}$ -values confirm that the membership degree of the allowable capacity, $\overline{\mu}_{C_a}(u)$, make sense. The mean and standard deviation of the interval-valued normal random variable $\overline{z}$ can be accordingly estimated as $\overline{m}$ =[-18.744,-8.853] and $\overline{\sigma}$ =[65.886,66.458] respectively. The fact that $\overline{m} \le \overline{0}$ clearly indicates the system requires PM. System data $\overline{t}_a$ can be used to fit Weibull distributions for further conventional reliability analysis.

6 Concluding Remarks

In this paper, we briefly discuss the concept of IVFS and argue its necessity to use IVFS idea for the modeling system reliability. We could simply use the method by Wu [] IVFS to conduct fuzzy inference on the system reliability directly. However, the virtual operational state of an operating system gives another inside of the operating system state. Using IVFS to analyze the system state seems more meaningful. For simplification reason, we skip quite many computation details by just refer to our previous work, Guo and Love [3]. As a matter of fact, it is more realistic to calculate the interval-valued membership grades and then use the logical function idea to have the IVFS membership grades for the system state.

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