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COMPARISONS OF EX POST EFFICIENT PORTFOLIOS GENERATED FROM SHARPE AND TROSKIE-HOSSAIN INDEX MODELS

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Abstract: In this paper we investigate the behavior of efficient frontier when the Troskie-Hossain covariance with highly correlated residuals is introduced under the Markowitz portfolio theoretical framework. The empirical evidences show that portfolios modeled with GARCH(1,1) components have the most efficient risk-return frontiers than these of state-space, ordinary least squares (OLS) and autoregressive moving average (ARMA) counterparts with the Troskie-Hossain covariance and Sharpe covariance. Our numerical evidences strongly suggested that the index models with Sharpe's covariance are consistently under-estimating the risks of the portfolios constituted of stocks from South African Stock market. On the contrary, Troskie-Hossain covariance provides a more realistic risk-return position of portfolios than that of Sharpe since it incorporates more information.

1. INTRODUCTION

Many portfolio optimization models investigated, particularly the Markowitz model and the Sharpe index models, are based on the first two moments of the random components with *uncorrelated* variance-covariance structure. In this paper, we relax this assumption and propose Troskie-Hossain models with the correlated variance-covariance structure. The *ex post efficient portfolio* investigations are carried on in terms of GARCH volatility modelling on the time-series based innovation processes. Numerical evidences show us exciting signals which will produce significant impacts on investment industry.

The paper is structured as follows. Section two is used to describe the classical portfolio selection problem, the efficient frontier concept in the discrete-time environment; the Sharpe Algorithm used for the frontier construction and then introduces the Sharpe Single Index model and its formulations. In section 3, we review the discrete-time stochastic volatility models, particularly GARCH(1,1) first and then investigate its various features. Section four presents a brief introduction into the extensions of the GARCH model, namely the EGARCH, C-GARCH, TARCH and finally the PARCH model. Then some classical portfolio formulations are presented. Section five concludes with an empirical study conducted on the South African stock market, the JSE, for a given portfolio of nine stock returns.

2. MODERN PORTFOLIO THEORY

2.1 Markowitz Mean-Variance Portfolio Selection Approach

A portfolio of p stocks is assumed, denoted by $\underline{r}^T = (r_1, r_2, \cdots, r_N)$ with the expected returns $\mathrm{E}[\underline{r}] = \underline{\mu}$. Assume that the covariances between returns of different returns of different shares are non-zero thus the covariance matrix of the stock returns is $\Xi = \mathrm{E}\left[(\underline{r}-\underline{\mu})(\underline{r}-\underline{\mu})^T\right]$ where $\sigma_{ii}=\sigma_i^2$ is the variance of the *i*'th stock and σ_{ij} the covariance between the *i*'th and the *j*'th stock. Let $\underline{w}^T = (w_1, w_2, \cdots, w_N)$ then the portfolio is $\underline{w}$

$$P = \underline{w}^T \underline{r} \text{ s.t. } \sum_{i=1}^{N} w_i = 1 \tag{1}$$

Thus we want to choose the weight vector $\underline{w}_i$ such that the expected return $\mathrm{E}[\underline{r}] = \underline{\mu}$ is a maximum but also at the same time that the risk or variance σ_p^2 is a minimum. The portfolio selection problem then becomes

$$\begin{cases} \max\limits_{\underline{w}} \left\{ \underline{w}^T \underline{\mu} \right\} \\ \min\limits_{\underline{w}} \left\{ \underline{w}^T \Xi \underline{w} \right\} \end{cases} \tag{2}$$

s.t.

$$\underline{w}^T \underline{1} = 1,\ w_i \in (0,1), \forall i = 1,2,\cdots,n$$

2.2 The Efficient Frontier

Let s_P^2 be the horizontal axis and E_k the vertical axis, the boundary line of all efficient portfolio pairs (s_P^2, E_k) is referred to as the *efficient frontier* for the portfolio *P* of given *N* assets. We then generate the Efficient Frontier and the investor takes a position somewhere on this frontier.

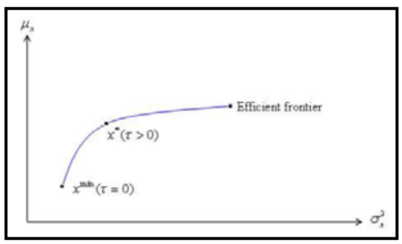

Figure 1. Efficient Frontier of a Given Portfolio *P*

2.3 The Sharpe Algorithm for Efficient Frontiers

Consider the Efficient Frontier for the *N* stocks. We draw a straight line through the frontier. Call this line Y = A+BX or μ_p=A+Bσ_p^2. Let the slope B=(1/φ) then the line parallel to the σ_p^2=X axis is where the slope B=0, that is φ=∞,and the line parallel to the μ_p =Y axis is the line where the slope B=∞ or φ=0. If we now take the line (A/B)=(1/B)μ_p - σ_p^2 or Z=φ μ_p - σ_p^2 and maximize A keeping B fixed, that is maximize Z keeping φ fixed will give us a point on the Efficient Frontier. Now by varying the slope B that is φ will generate the Efficient Frontier. Thus our problem reduces to Max(Z) = μ_p - σ_p^2 =φW'μ-W'ΣW. We vary φ from (0,∞) to generate the Efficient Frontier.

2.4 The Sharpe Single Index Model

Let the return (log) of particular stock be $R_t = log(P_t) – log(P_{t-1})$, and $I_t = log(I_t) – log(I_{t-1}), t=1,…,T$ be the return of the market proxy. Then the market model is assumed to be $R_t = \alpha + \beta I_t + et, t=1,…,T$ for a single stock R_t

$$\begin{aligned} E(e_t^2) &= \sigma_e^2 \\ E(e_t e_s) &= 0, \quad t \neq s = 1,…,T \end{aligned} \tag{3}$$

Let the *i*th stock be $R_{it} = \alpha_i + \beta_i I_t + e_{it}$, $i=1,…,N;\ t=1,…,T$, where all stocks are regressed on the same single index *I*, then

$$E(e_{it}^2) = \sigma_{ei}^2$$
$$E(e_{it}e_{is}) = 0, \quad t \neq s=1,\ldots,T,$$
$$E(e_{it}I_t) = 0, \quad t=1,\ldots T, \qquad (4)$$
$$E(e_{it}e_{jt}) = 0, \quad t=1,\ldots,T.$$

The variance is $var(R_i) = var(\alpha_i + \beta_i I + e_i) = \beta_i^2\sigma_I^2 + \sigma_{ei}^2 = \sigma_{ii} = \sigma_i^2$ and the covariance is $cov(R_iR_j) = \beta_i\beta_j\sigma_I^2$. Then the portfolio selection problem then becomes

$$\max Z = \varphi \underline{w}^T \underline{\mu} - \underline{w}^T \Xi \underline{w}$$
$$\text{s.t.} \qquad (5)$$
$$\underline{w}^T \underline{1} = 1$$

2.5 The Sharpe Multiple Index Model

The multi - index model can be written as

$$R_{it} = \alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + \ldots + \beta_{iM}I_M + e_{it}$$
$$i = 1,\ldots,N;\ t = 1,\ldots,T \qquad (6)$$

with the following assumptions

$$E(e_{it}^2) = \sigma_{ei}^2$$
$$E(e_{it}\,e_{is}) = 0, \quad t \neq s=1,\ldots,T$$
$$E(e_{it}\,I_{jt}) = 0, \quad j=1,\ldots,M, t=1,\ldots T \qquad (7)$$
$$E(e_{it}\,e_{it}) = 0, \quad t=1,\ldots,T$$
$$E(I_{jt}I_{kt}) = c_{jk}, \quad j,k=1,\ldots,M$$

The Sharpe formulation for multiple index is then (minimize instead of maximize), Min $Z = -\Lambda\mu_p + \sigma_p^2$ subject to

$$\beta_{p1} = \sum_{i=1}^{p} w_i\beta_{i1},\ \beta_{p2} = \sum_{i=1}^{p} w_i\beta_{i2},\ldots,\ \beta_{pM} = \sum_{i=1}^{p} w_i\beta_{iM} \text{ and } \sum_{i=1}^{p} w_i = 1 \qquad (8)$$

and any other equality, in-equality constraints or bounds. Then objective function becomes

$$\min Z = -\lambda\mu_P + \sigma_P^2 + \sum_{j=1}^{M}\lambda_j\left(\beta_{Nj} + \sum_{i=1}^{N} w_i\beta_{ij}\right) + \lambda_f\left(1 - \sum_{i=1}^{N} w_i\right) \qquad (9)$$

The bounds in this case are $0 \leq w_i \leq 1$ so the first stock that enters the efficient frontier is the one with the largest return E_i.

3. TROSKIE-HOSSAIN INDEX MODELS

3.1 The Troskie-Hossain Single Index Model

Recall that the Sharpe single index model assumes

$$\operatorname{cov}(R) = \sigma_I^2 \beta\beta' + \begin{pmatrix} \sigma_{e1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{e2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \sigma_{ep}^2 \end{pmatrix} \tag{10}$$

Troskie-Hossian treatment relaxes this assumption and allows the correlation between stocks. They use the time-series based modelling innovation process to extract the variance-covariance structure. Let

$$\hat{\mathbf{E}} = \begin{pmatrix} \hat{e}_{11} & \hat{e}_{12} & \dots & \hat{e}_{1N} \\ \hat{e}_{21} & \hat{e}_{22} & \dots & \hat{e}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{e}_{p1} & \dots & \dots & \hat{e}_{pN} \end{pmatrix} \tag{11}$$

then

$$\hat{\Omega} = \frac{1}{N - M - 1} \hat{E} \hat{E}' \text{ and } \Phi = \sigma_I^2 \beta\beta' + \Omega \tag{12}$$

3.2 The Troskie-Hossain Multiple Index Model

The multi-index model modification can be carried on in similarly way. The Indices are assumed to be dependent with covariances given by c_{jk}. Let $E_i = E(R_i) = \alpha_i + \beta_{i1}\mu_1 + \ldots + \beta_{iM}\mu_M$, i= 1,…,p. Then $R_t = \alpha + \beta I_t + e_t$ with $E(R_t) = \alpha + \beta\mu_I$. The covariance matrix of R_t is then

$$\begin{aligned} cov(R_t) &= E[R_t - E(R_t)][R_t - E(R_t)]' = E[\alpha + \beta I_t + e_t - \alpha - \beta\mu_I][\alpha + \beta I_t + e_t - \alpha - \beta\mu_I]' \\ &= \beta C\beta' + \Omega = \Phi \end{aligned} \tag{13}$$

For portfolio $P = W^T R$, $E_p = E(P) = W^T(\alpha + \beta\mu_I)$ and $\sigma_p^2 = var(P) = W^T\Phi W$. The portfolio problem is then Min $Z = -\Lambda E_p + \sigma_p = -\Lambda W'(\alpha + \beta\mu_I) + W'\Phi W$. Then the problem can be solved using calculus by implementing the multiple index simultaneous equations with Φ substituted for Σ. Our estimates would be

$$\begin{aligned} \hat{E}_p &= W'\hat{E} \\ \hat{\Omega} &= \hat{E}\hat{E}^T / (N - M - 1) \end{aligned} \tag{14}$$

4. EMPIRICAL INVESTIGATIONS

4.1 Troskie-Hossain Single Index versus Sharpe Single Index

For this empirical analysis, the data used were the monthly log returns for the stocks Anglos, Jdgroup, Pick and Pay, Remgro, SA-Eagle, Sappi, Sasol, Tigerbrands and Tongaat represented as R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8 and R_9 respectively. The modelling was done over the period July1988-February 2005.

Numerical evidences show that GARCH(1,1) is superior. Comparing the remaining 4 techniques, ARMA, OLS, MLE with each other, it is established that OLS is the worst, then the State Space and ARMA model outperforming it marginally in the respective order. The MLE and OLS models yield similar estimates of β hence the efficient frontiers are very close to each other.

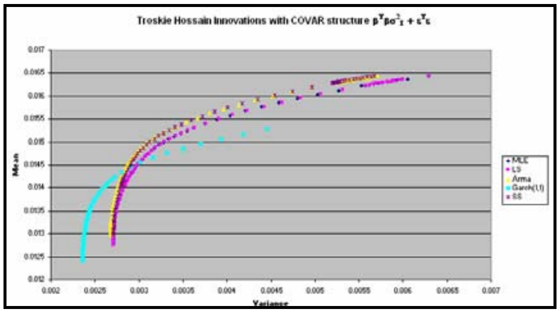

Figure 2. Efficient Frontiers for the Troskie-Hossain Single Index Model

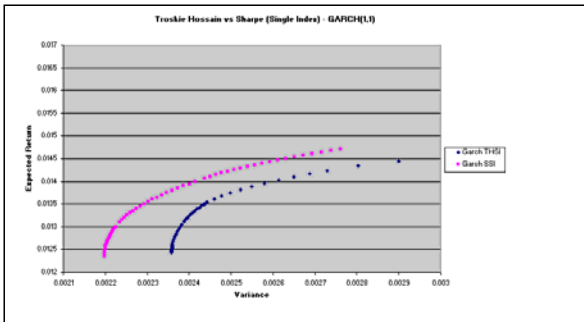

Figure 3. GARCH SSI versus GARCH THSI

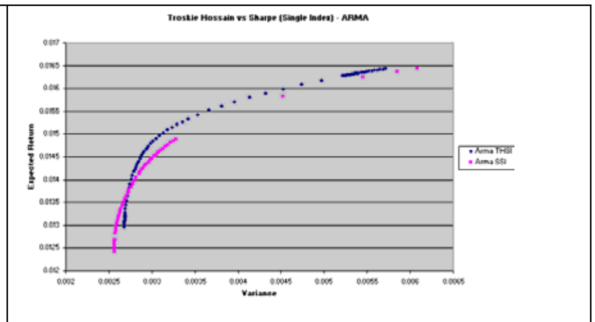

Figure 4. ARMA SSI versus ARMA THSI

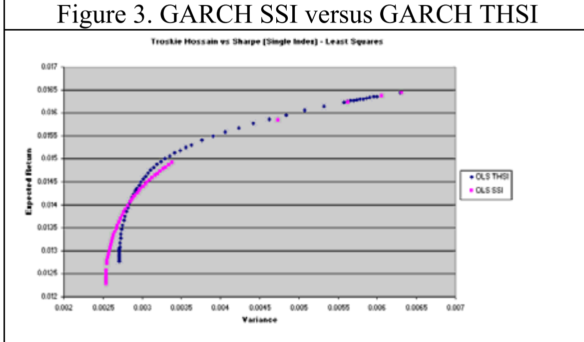

Figure 5. OLS SSI versus OLS THSI

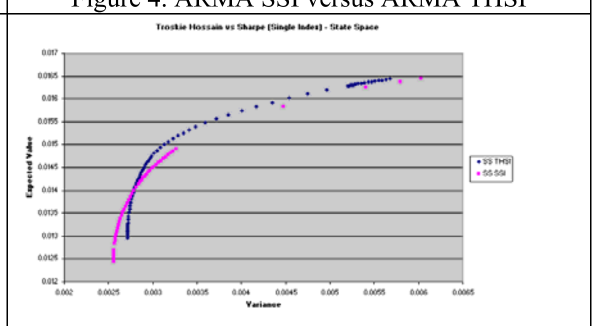

Figure 6. SS SSI versus SS THSI

The GARCH(1,1) Sharpe frontier is to the left of the Troskie-Hossain GARCH(1,1) frontier, hence it is obvious from the above intuition that Sharpe is underestimating risk at every level of return when compared to Troskie-Hossain models.

4.2 Troskie-Hossain Multiple Index versus Sharpe Multiple Index

The objective will be to construct efficient frontiers for GARCH(1,1), ARMA, OLS and State space multiple index models implementing the Troskie Hossain Innovation for the covariance structures of the Indices and the disturbance terms e_{it}.

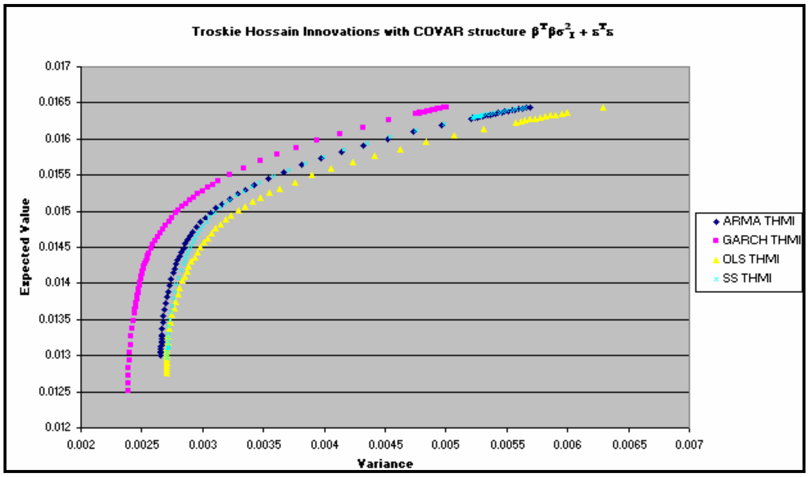

Figure 7. Efficient Frontiers for the Troskie Hossain Multiple Index

Like in the single index scenario, the efficient frontiers observed under the Troskie-Hossain multiple index context have produced consistent results with that of their Troskie-Hossain single index counterparts. It is empirically established that GARCH(1,1) is the most superior, then ARMA, State Space follows next, and finally then OLS. Next we compare the Sharpe frontiers with the Troskie-Hossain frontiers

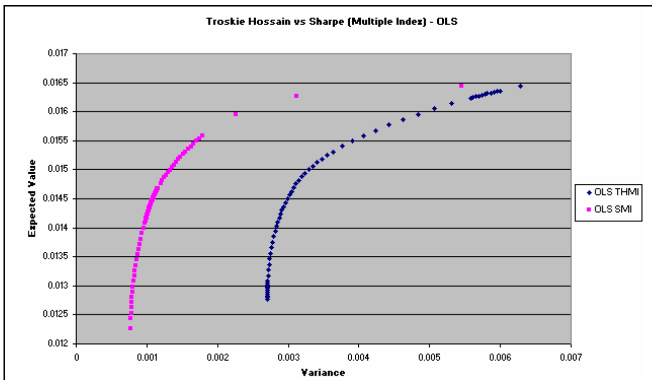

Figure 8. OLS SMI versus OLS THMI

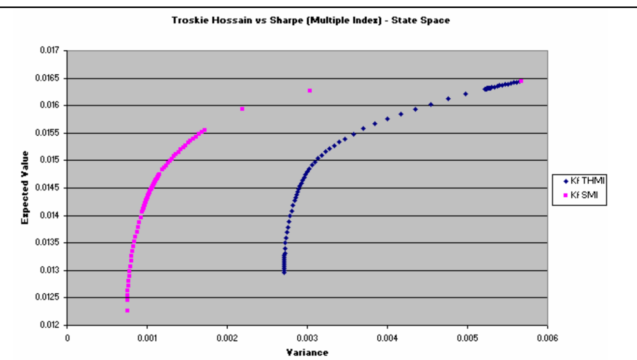

Figure 9. SS SMI versus SS THMI

In the multiple index setting this is even more pronounced due to there being more indices now, hence the SMI model ignores more information than before thus further understating the current risk level. As a result, the Troskie-Hossain multiple index and Troskie-Hossain single index models are a more accurate perception of the portfolio's true risk-return profile in the market.

5. CONCLUSIONS

Given the primary findings above in both empirical sections, it can be concluded that Sharpe's Single and Multiple index models are restricted in terms of information compared to that of the Troskie-Hossain counterpart. Furthermore, it has been empirically established that Sharpe is under-estimating the risk profile of portfolios in the South African context. Hence the Sharpe model will actually result in investors taking on more risk than they are actually supposed to and damage the investment industry.

6. REFERENCES

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