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STATISTICAL MODELING OF SYSTEM AGES

R. Guo[1] and C. E. Love[2]

[1]Department of Statistical Sciences
University of Cape Town
Private Bag, Rondebosch 7700
Cape Town, South Africa
Corresponding author's e-mail: rguo@stats.uct.ac.za

[2]Faculty of Business Administrations
Simon Fraser University
Burnaby, B.C. V5A 1S6, Canada

Abstract: System age is one of the key parameters in system reliability modeling and maintenance in policy decision-making. However system age, particularly found in complex systems, is often unobservable. It is well recognized in the engineering literature that a system's chronological age is often not a true reflection of its useful age in terms of system reliability. As a consequence, different system age models have been proposed such as Kijima's virtual age, intrinsic age models to name only two. Although such concepts have played an important role in system reliability modeling and maintenance optimization, they are in nature mathematical and impose an a *priori* model on the process. More seriously parameters of such models are virtually impossible to statistical estimate. It has also be well recognized that while there often exists extremely rich operating and maintenance records for such systems, the age modeling being preformed has largely ignoring this readily available information.

In this paper, we propose a framework which decomposes the system age process into two sub-processes; age recovery processes which are negative-valued stochastic processes and aging (due to wear-out and damages) processes which are monotone increasing valued stochastic processes. Then we analyze the basic properties of these age recovery and aging sub-processes and explore the methodology to link them to the corresponding system operating/maintenance history for establishing a statistically estimable system age.

1. INTRODUCTION

System age is understood to be one of the key parameters in system reliability modelling and maintenance policy decision-making. However system age, particularly in complex systems, is often not a directly observable or recordable factor notwithstanding that a system always reveals its age information one way or another indirectly. Such indirect information is observable and recordable. It is recognized in the engineering literature that a system's *chronological age* is often not a true reflection of a system's age for the purposes of reliability prediction. It is obvious that the system reliability, $R(t)$, does not exactly depend upon the *chronological* age t but its *true* age $A(t)$. The immediate implication is that the classical definition of reliability will generate error in modelling and the definition of system reliability should be modified to:

$$R(A(t)) = \exp\left(-\int_0^t h(A(u))\left(\frac{dA}{du}\right)du\right) \qquad 1$$

where $A(t)$ is the true system age at time t and $h(x)$ *is* the hazard function. Furthermore one observes in engineering a *substitute* age is often utilized for a particular system. For example, car mileage is substituted for a car's chronological age believing it to be more indicative of the system's true age. In order to recognize such intrinsic characteristics of a system age (like the *bioclock* identified in human beings), researchers in operational research and particularly in reliability engineering have proposed different age concepts (e.g., *intrinsic* age and *virtual* age (Kijima 1989)). The common feature of this various age models is the idea of *jump* behaviours.
Although these various concepts have played an important role in system reliability modelling and maintenance optimization, they are in reality merely mathematical constructs imposing an *a* priori model on the system. The virtual age of a system is non-estimable statistically, e.g., Guo, Ascher and Love (2000, 2001). Furthermore we see the interesting dilemma that there exists huge data banks of system operating and maintenance records while observing researchers

proposing age models largely ignoring this information. It is certainly expeditious to ignore such data and often it is a major undertaking to collect it. However such expediency is no doubt often driven by a failure to understand just how these data could be fully utilized for reliability analysis purposes.
In this paper, based on observing the fluctuations in system performance from time to time as well as identifying possible jumps, we propose decomposing the system age path into two types: a continuous (diffusion) one and a discrete (jump) one.

$$A(t) = A^c(t) + A^j(t). \tag{2}$$

The jump path could be further decomposed into two sub-processes: an age-decrement process (due to repair and maintenance) and an age-increment process (due to shocks and damages).

$$A^j(t) = A^i(t) + A^d(t). \tag{3}$$

Both of the age increment and decrement processes are compound counting processes in nature.

$$A^i(t) = \sum_{l=0}^{N_1(t)} Y_l^i \text{ and } A^d(t) = \sum_{k=0}^{N_2(t)} Y_k^d. \tag{4}$$

We then establish a system age dynamic in terms of stochastic differential equation (SDE) with respect to a martingale or semi-martingale process. In Section 2 of this paper we provide a brief review of semi-martingale stochastic calculus. In Section 3 the SDE for our age dynamics is discussed. That is, in these two sections we analyze the basic properties of the age increment and decrement sub-processes and explore the methodology to link them to the corresponding system operating/maintenance history to allow a statistically estimable system age. In Section 4 we present a Bayesian estimation of age increments and in Section 5 we discuss a fuzzy evaluation of age decrements. Finally we give a few remarks on the statistical modeling of system age and point to further research directions.

2. SEMI-MARTINGALE CALCULUS

Definition 1 A stochastic process $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ is called a semi-martingale if it has the following decomposition:

$$X_t = X_0 + M_t + A_t \text{ a.s. } [P] \tag{5}$$

where $M \in \mathcal{M}^{loc}(P), A \in \mathcal{V}(P)$, *and* $M_0 = A_0 = 0$.

Lemma Let $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ be a semi-martingale. Then X admits a unique representation:

$$X_t = X_0 + M_t + \alpha_t + \sum_{s \leq t} \Delta X_s \vartheta(|\Delta X_s| > 1) \tag{6}$$

or

$$X_t = X_0 + M_t^{(c)} + \alpha_t + \sum_{s \leq t} \Delta X_s \vartheta(|\Delta X_s| > 1) + M_t^{(d)} \tag{7}$$

where $M \in \mathcal{M}^{loc}$, $M = M^{(c)} + M^{(d)}$, and $\alpha \in \alpha^{loc}(P) \cap \mathcal{P}$ and

$$\Delta X_s = X_s - X_{s-}. \tag{8}$$

Definition 2 Let X be a semi-martingale with decomposition:

$$X_t = X_0 + M_t + A_t$$ 9

and H is a predictable locally bounded process. Then the stochastic integral of X is :

$$H \cdot X \triangleq H \cdot M + H \circ A$$ 10

no matter what the representation of X is.

for $\forall t \geq 0$, and the infinite product is convergent *a.s.* (here X^c is the continuous component of X and $\Delta X_s = X_s - X_{s-}$.

3. A STOCHASTIC DIFFERENTIAL EQUATION FOR SYSTEM AGE DYNAMICS

Therefore, $\{A(t), t \in \mathbb{T}\}$ is assumed to be a semi-martingale process satisfying the following stochastic differential equation:

$$dA(t) = \zeta(A(t), t, \alpha)dt + \eta(A(t), t, \beta)dW(t) + \sum_{i=1}^{2} \theta_i(A(t), t, \gamma_i)dZ_i(t), t \geq 0,$$ 11

$$A(0) = a$$

where $\{W(t), t \in \mathbb{T}\}$ is the standard Wiener process and $\{Z_1(t), t \in \mathbb{T}\}$ and $\{Z_2(t), t \in \mathbb{T}\}$ are mutually uncorrelated compound Poisson processes independent of $\{W(t), t \in \mathbb{T}\}$. Under regular conditions the solution to the stochastic differential equation exists and unique.

Notice that by using the superposition theorem of Compound Poisson processes the SDE

$$\begin{cases} dA(t) = \theta A(t)dt + \sigma A(t-)dW(t) + A(t-)\sum_{k=1}^{2} \gamma_k dA_k^j(t) \\ A(0) = a \end{cases}$$ 12

becomes

$$\begin{cases} dA(t) = \theta A(t)dt + \sigma A(t-)dW(t) + A(t-)dA^j(t) \\ A(0) = a \end{cases},$$ 13

which has a unique closed form of solution.

A simple system age model here merely means that the age process path is decomposed into a continuous one and a discrete (jump) one.

$$A(t) = A^c(t) + A^j(t),$$ 14

so that,

$$\begin{cases} dA(t) = \theta A(t)dt + \sigma A(t-)dW(t) + A(t-)dA^j(t) \\ A(0) = a \end{cases}$$ 15

Where:

$$A^j(t) = \sum_{l=1}^{M(t)} Y_l$$ 16

with $\{N(t), t \in \mathbb{T}\}$ being a Poisson process having intensity $\lambda > 0$ and Y_l, $l \geq 1$ are *i.i.d.* random variables $\left(Y_l > -1, l \geq 1\right))$ independent of $\{N(t), t \in \mathbb{T}\}$, and both $\{N(t), t \in \mathbb{T}\}$ and $\{Y_l, l \geq 1\}$ being independent of $\{W(t), t \in \mathbb{T}\}$. The unique solution to the SDE is:

$$A(t) = a\exp\left(\left(\theta - \frac{1}{2}\right)t + W(t)\right)\prod_{l=1}^{N(t)}(1 + Y_l) \qquad 17$$

When $\sigma = 1$.

4. A BAYESIAN ESTIMATION OF SYSTEM AGE INCREMENTS

A critical step of age modeling is to estimate system age increments and decrements respectively. Notice that shocks (in terms of damage and wear) which are often observable or at least partially detectable (in terms of systems containing sensors for measurement purposes are treated as situations demanding maintenance attention. The physical damage measure process $\mathsf{X} = \{X_t, t \in \mathbb{R}^+\}$ is in nature a compound process:

$$X_t = \alpha_0 + \sum_{i=1}^{N_t} Y_i, \qquad 18$$

where $\{N_t, t \in \mathbb{R}^+\}$ is a counting process. In an aeroplane investigation, a basic shock damage model was investigated by Esary, Marshall and Proschan (1973). The physical process descriptors were first identified and quantified as crack increments and shock occurrences. With the notation of

$$A(\alpha, k) = \alpha + \sum_{i=1}^{k} Y_i \qquad 19$$

being the crack size found at inspection when the state is (α, k). The distribution of $\Delta k(\alpha, k)$, the number of shocks during a flight, is a Poisson $(\lambda s(\alpha, k)) + 1$ random variable. "Since the function $s(\alpha, k)$ offers more degree of freedom than we are likely to need, there will be considerable latitude in defining this function so as to fit the data." Let the critical crack size ξ be a random variable so that if the crack size is larger than ξ, then a failure occurs. The probability of failure in flight when the state at the beginning of the flight is (α, k) :

$$p(\alpha, k) = \Pr\left[A(\alpha, k) + \sum_{i=k+1}^{k+\Delta k(\alpha,k)} Y_i > \xi\right]. \qquad 20$$

Shock models in nature lead to the development of compound (Poisson) processes (e.g., Barbour, Chryssaphinou and Malgorzata, 1996). For a repairable system, the time from which a defect can be identified at an inspection to the random time that the defect causes a system failure is said to be a *delay time* (Christer and Wang (1995a, 1995b) and Christer, Wang, Baker and Sharp (1995)). As Wang (1997) pointed out "the delay time concept defines a two stage stochastic process where the first stage is the initiating phase of a defect and the second is the stage where the defect leads to a failure." Assuming that the defects occurred and can be identified as following a point process $\{N_t, t \in \mathbb{R}^+\}$, with the i^{th} delay time being Y_i, $i = 0, 1, 2, \ldots$, and the *inter-arrival time* between the i^{th} and $(i+1)^{st}$ defects being X_t , $\{X_i, i = 1, 2, \cdots\}$, then we have a delay time

$$Y_{N_i} = L - \sum_{i=1}^{N_i} X_i, \quad (21)$$

where L is the lifetime starting as a defect-free system. Then the delay time is again a compound process similar to that of cumulative shock damage processes which can be readily used to estimate the age increments.

5. AGE RECOVERY SUB-PROCESSES

Guo and Love (2003) demonstrated that the repair or maintenance impacts to an imperfectly repaired system might be evaluated by applying fuzzy mathematic-statistical methodology. In the same way, a fuzzy evaluation of system age deductions due to repair or maintenance can be quantified and thus the age decrement at time t, at which repair or maintenance is carried out can be statistical evaluated.

Sample membership function from linguistic evaluation of repair or maintenance impacts established by using the tone operator's quantitative value and the corresponding sample membership degree offered in Chen (1998) offers a way to calculate the relative membership grade of the linguistic PM effect evaluation.

Tone	same	slightly	little	more	obviously	significantly	fully	absolutely	tremendously	extreme	absolutely superior
quantity	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1
μ^r	1.0	0.818	0.667	0.538	0.429	0.333	0.25	0.176	0.111	0.053	0

TABLE 1: Tone operator, Quantitative value and sample membership grade

Therefore the sample membership grade of the impacts from repair and maintenance could be extracted. The next step is to evaluate the membership grades based on the system operating data because the system operating or performance (after repair or maintenance) further reveals the age impact from repair or maintenance. Let the **X** be the characteristic value matrix of the n operating indices of the m repairs or maintenances and x_{ij} is the characteristic value of the j^{th} operating index in the i^{th} repair or maintenance. In terms of the normalization formula, i.e., the formula for the sample membership grade:

$$\mu_{ij}^r = \frac{x_{ij} - \min_i \{x_{ij}\}}{\max_i \{x_{ij}\} - \min_i \{x_{ij}\}} \quad (22)$$

which is suitable for the case where the larger the characteristic value the better. However, one suspects the cost of repair or maintenance impacts on system operating is time-lagged. For a simplified evaluation, the PM cost impact is only effective on the next PM. Therefore the value of the relative membership grade of the cost of a PM is one row-shifted to reflect the engineering justification.

Finally, let us evaluate the sample membership grade of the fuzzy set of the system age. Notice that we do not have the full information on all aspects of fuzzy reliability. We only have two aspects and even then only partial information. Therefore the logical function behind the sample membership function of age deduction is

$$\widetilde{\Delta A}^d(t) = \beta\left((\widetilde{Cost}(t), \widetilde{PM}, \underline{\widetilde{X}}\right), \quad (23)$$

were $\underset{\sim}{\tilde{X}}$ reflects partial information of a system's operating environment and $\widetilde{Cost}$ and $\widetilde{PM}$ reflects the repair and maintenance impacts on the operating system. Based on the age deduction logical function, the sample membership function for age decrements, $\mu^{r}_{\widetilde{\Delta A}(t)}(\cdot)$, now can be calculated and therefore the fuzzy set for age decrement, $\widetilde{\Delta A}^{d}(t)$, at time t is determined.

Because the fuzzy semi-martingale theory is not available yet, thus it is necessary to use *defuzzification* to convert the age decrement into a single number as an estimate.

6. CONCLUDING REMARKS

In this paper, we propose an age model in terms of semi-martingale processes theory. However, the key issue of the statistical analysis of the age process is still dependent upon the estimation of the jump process which is decomposed into the age increments and age decrements respectively. We develop a proposition of the superposition of compound Poisson processes and its corollary which decomposes a compound Poisson process into up-jump and down-jump sub-processes respectively. Therefore, we then have an ability to estimate the age increments by Bayesian statistical methodology and estimate the age decrements in terms of fuzzy methodology. With the increment information the parameters in the age process can be estimated and the dynamics of system age can be fully explored.

Since the age dynamics obtained is a Markov process and its expectation is readily established, the optimal maintenance could be easily determined. However, more efforts are needed to refine the age stochastic processes for reliability engineering applications.

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